

PROOF:

a) Take $f, g \in G$. Then, f and g are bijective. According to Proposition 9, $g \circ f$ is also bijective i.e., $g \circ f \in G$. G is closed.

b) According to Proposition 8 $\forall f, g, h \in G, h \circ (g \circ f) = (h \circ g) \circ f$. G is associative.

c) According to Proposition 7 $\forall f \in G, f \circ I_X = I_X \circ f = f$. Since I_X is bijective, then $I_X \in G$. So, I_X is the identity element of G i.e., G has identity element.

d) According to Proposition 12, $\forall f \in G, \exists f^{-1} \in G: f^{-1} \circ f = f \circ f^{-1} = I_X$ because each function in G is bijective.

Now, we show that G is commutative if and only if $|X| \leq 2$:

" \Rightarrow ": Assume that $|X| > 2$. Let's show that G is not commutative. Since $|X| > 2$, then G has at least three distinct elements. Let a, b and c denote three distinct elements in G and define the functions $f, g: X \rightarrow X$ as follows:

$$f(x) = \begin{cases} b, & x = a \\ c, & x = b \\ a, & x = c \\ x, & x \in X \setminus \{a, b, c\} \end{cases} \quad g(x) = \begin{cases} a, & x = a \\ c, & x = b \\ b, & x = c \\ x, & x \in X \setminus \{a, b, c\} \end{cases}$$

It is obvious that f and g are bijective. So, $f, g \in G$.

Since $(g \circ f)(b) = g(f(b)) = g(c) = b$ and $(f \circ g)(b) = f(g(b)) = f(c) = a$, (i.e., $(g \circ f)(b) \neq (f \circ g)(b)$) then $g \circ f \neq f \circ g$. Consequently, G is not commutative.

" \Leftarrow ": Assume that $|X| \leq 2$. Then, $|X| = 1$ or $|X| = 2$.

If $|X| = 1$, then $X = \{a\}$. (Where a is any object) In this case, there is only one function from $\{a\}$ to $\{a\}$. This is I_X being the identity function of X . According to this, $G = \{I_X\}$. Since every group with one element is commutative, then G is also commutative.

If $|X| = 2$, then $X = \{a, b\}$. (Where a and b are objects) There are two bijection from $\{a, b\}$ to $\{a, b\}$. One of them is I_X and the other is the function $f_0 = \{(a, b), (b, a)\}$. (I.e., f_0 is the function satisfying $f_0(a) = b$ and $f_0(b) = a$) According to this, $G = \{I_X, f_0\}$. Since every group with two elements is commutative, then G is also commutative.