PROBLEMS AND SOLUTIONS ON THE SET THEORY

1. Prove the following propositions:

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

c) $A \Delta B = (A \cup B) \setminus (A \cap B)$,

d) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

e) $A \Delta \varnothing = A$,

f) $A \setminus B = B^C \setminus A^C$,

g) $A^C \Delta B^C = A \Delta B$,

h) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

(Where A, B, C are sets.)

2. Prove the following propositions:

a) $A \cup B = B \Leftrightarrow A \subset B \Leftrightarrow A \cap B = A \Leftrightarrow A \setminus B = \varnothing$,

b) $A \Delta B = \varnothing \Leftrightarrow A = B$.

(Where A, B are sets.)

3. Prove that the power set of a set having n elements has 2^n elements i.e.,

$|X| = n \Rightarrow |\mathrm{P}(X)| = 2^n$

SOLUTIONS

1. a) $x \in A \cup (B \cap C) \Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

$\Leftrightarrow x \in A \cup B \wedge x \in A \cup C \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

b) $\left[A \cap (B \cup C)\right]^C = A^C \cup (B \cup C)^C = A^C \cup (B^C \cap C^C) = (A^C \cup B^C) \cap (A^C \cup C^C)$

$= (A \cap B)^C \cap (A \cap C)^C = \left[(A \cap B) \cup (A \cap C)\right]^C$

We obtain $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ by using (a), the property $\left(A^C\right)^C=A$ and De Morgan's law.

c)

$$A\Delta B=(A\setminus B)\cup(B\setminus A)=\left(A\cap B^C\right)\cup\left(B\cap A^C\right)=\left[\left(A\cap B^C\right)\cup B\right]\cap\left[\left(A\cap B^C\right)\cup A^C\right]$$

$$=\left[(A\cup B)\cap\left(B^C\cup B\right)\right]\cap\left[\left(A\cup A^C\right)\cap\left(A^C\cup B^C\right)\right]=\left[(A\cup B)\cap E\right]\cap\left[E\cap(A\cap B)^C\right]$$

$$=(A\cup B)\setminus(A\cap B)$$

d) $(A\Delta B)\Delta C=\left[\left(A\cap B^C\right)\cup\left(B\cap A^C\right)\right]\Delta C$

$$=\left[\left[\left(A\cap B^C\right)\cup\left(B\cap A^C\right)\right]\cap C^C\right]\cup\left[C\cap\left[\left(A\cap B^C\right)\cup\left(B\cap A^C\right)\right]^C\right]$$

$$=\left[\left(A\cap B^C\cap C^C\right)\cup\left(A^C\cap B\cap C^C\right)\right]\cup\left[C\cap\left(A^C\cup B\right)\cap\left(A\cup B^C\right)\right]$$

$$=\left[\left(A\cap B^C\cap C^C\right)\cup\left(A^C\cap B\cap C^C\right)\right]\cup\left[\left[\left(A^C\cap C\right)\cup(B\cap C)\right]\cap\left(A\cup B^C\right)\right]$$

$$=\left(A\cap B^C\cap C^C\right)\cup\left(A^C\cap B\cap C^C\right)\cup\left(A^C\cap B^C\cap C\right)\cup(A\cap B\cap C).$$

We have $(A\Delta B)\Delta C=\left(A\cap B^C\cap C^C\right)\cup\left(A^C\cap B\cap C^C\right)\cup\left(A^C\cap B^C\cap C\right)\cup(A\cap B\cap C)$. In the expansion $\left(A\cap B^C\cap C^C\right)\cup\left(A^C\cap B\cap C^C\right)\cup\left(A^C\cap B^C\cap C\right)\cup(A\cap B\cap C)$, A, B and C can be replaced each other because union and intersection operations are commutative. I.e., the equations $(A\Delta B)\Delta C=(B\Delta A)\Delta C=(C\Delta A)\Delta B=(A\Delta C)\Delta B=(B\Delta C)\Delta A=(C\Delta B)\Delta A$ are provided. As well the symmetric difference operation is commutative, the equations $(A\Delta B)\Delta C=(B\Delta A)\Delta C=(C\Delta A)\Delta B=(A\Delta C)\Delta B=(B\Delta C)\Delta A=(C\Delta B)\Delta A$ $=C\Delta(A\Delta B)=C\Delta(B\Delta A)=B\Delta(C\Delta A)=B\Delta(A\Delta C)=A\Delta(B\Delta C)=A\Delta(C\Delta B)$ are true. Finally, $(A\Delta B)\Delta C=A\Delta(B\Delta C)$ q.e.d.

e) $A\Delta\varnothing=(A\cup\varnothing)\setminus(A\cap\varnothing)=A\setminus\varnothing=A$

f) $B^C\setminus A^C=B^C\cap\left(A^C\right)^C=B^C\cap A=A\cap B^C=A\setminus B$

g) $A^C\Delta B^C=(A^C\setminus B^C)\cup(B^C\setminus A^C)=(B\setminus A)\cup(A\setminus B)=A\Delta B$

h) $(x,y)\in(A\cup B)\times C\Leftrightarrow x\in A\cup B\wedge y\in C\Leftrightarrow(x\in A\vee x\in B)\wedge y\in C$

$(x\in A\wedge y\in C)\vee(x\in B\wedge y\in C)\Leftrightarrow(x,y)\in A\times C\vee(x,y)\in B\times C$

$\Leftrightarrow(x,y)\in(A\times C)\cup(B\times C)$

2. **a)** We denote,

(i) $A \cup B = B$,

(ii) $A \subset B$,

(iii) $A \cap B = A$,

(iv) $A \setminus B = \varnothing$.

(i)$\Rightarrow$(ii): We assume $A \cup B = B$. Then $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in B \Rightarrow A \subset B$.

(ii)$\Rightarrow$(iii): It is clear that $A \cap B \subset A$. Let's prove its inverse. By taking intersection of both sides of $A \subset B$, we have $A \cap A \subset A \cap B \Rightarrow A \subset A \cap B$. Consequently, $A \cap B = A$.

(iii)$\Rightarrow$(iv): We assume the contrary i.e., $A \setminus B \neq \varnothing$. Then, there exists a element providing $x \in A$ and $x \notin B$. Because $A \cap B = A$, we obtain the contradiction $x \in A \Rightarrow x \in A \cap B \Rightarrow x \in B$. Then, the assumption is false i.e., $A \setminus B = \varnothing$.

(iv)$\Rightarrow$(i): $A \cup B = (A \setminus B) \cup B = \varnothing \cup B = B$.

b)

$$A \Delta B = \varnothing \Leftrightarrow (A \setminus B) \cup (B \setminus A) = \varnothing \Leftrightarrow A \setminus B = \varnothing \wedge B \setminus A = \varnothing \Leftrightarrow A \subset B \wedge B \subset A \Leftrightarrow A = B$$

3. For $0 \leq k \leq n$, the number of all the subsets with k elements is $\binom{n}{k}$ because we choose any k elements in the n elements. Then the number of all the subsets of X is $\sum_{k=0}^{n} \binom{n}{k}$. By using the binomial expansion, we have $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$.